Ionospheric Absorption

Prepared by Forrest Foust
Stanford University, Stanford, CA

IHY Workshop on
Advancing VLF through the Global AWESOME Network
VLF Injection Into the Magnetosphere

- Earth-based VLF transmitters can inject wave energy into the magnetosphere by transmission through the lower ionosphere.
- Understanding the processes of transmission and loss is an important part of modeling the interaction of this wave energy with energetic particles in the radiation belts.

VLF injection and interaction

Satellite observations of earth-based VLF transmitters
Charged particles in magnetized plasmas naturally undergo gyromotion, or circular motion about magnetic field lines, and forced motion under the influence of a time-varying electric field.

This complex motion of charges gives rise to secondary currents.

These currents, in turn, modify the properties of electromagnetic waves propagating in a plasma.

Left: trajectories of electrons under the influence of both gyromotion and forced motion.

Extraordinary mode - forced motion in the same direction as natural gyromotion.

Ordinary mode - forced motion in a direction opposite to the natural gyromotion.
Energy can be lost in plasmas through *collisions*, which converts ordered motion of charges into disordered motion (heating).

**Two classes of collisions are important in most plasmas:**
- Electron-neutral collisions - an electron collides with a neutral molecule and scatters off at some angle.
- Electron-ion collisions - an electron scatters off an ion through the interaction of Coulomb forces.

Electron-neutral collisions are most important in partially ionized gases such as in the earth’s ionosphere.

Right: important types of collisions.

In an elastic collision, the electron still has some momentum after the collision.

In an inelastic collision, the electron loses all of its momentum.
Due to the large number of particles, plasmas are most commonly treated as a fluid coupled with Maxwell’s equations and an additional Lorenz force term.

The continuity condition (conservation of mass) is:

\[
\frac{\partial N_e}{\partial t} + \nabla \cdot [N_e u_e] = 0
\]

Conservation of momentum is:

\[
m_e N_e \left[ \frac{\partial u_e}{\partial t} + (u_e \cdot \nabla) u_e \right] = -\nabla p_e + q_e N_e (E + u_e \times B) - m_e N_e \nu u_e
\]

Collisional losses are modeled as a simple linear friction term.

It is common to drop the convective and pressure terms, yielding the “linearized cold plasma” equation:

\[
m_e N_e \frac{\partial u_e}{\partial t} = q_e N_e (E + u_e \times B) - m_e N_e \nu u_e
\]
Current and charge are related to the number density and electron velocity:

\[
J = N_e q_e u_e \\
\rho = N_e q_e
\]

Substituting, we can rewrite the fluid equations as relationships between \(J\), \(E\), and the charge density:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \\
J = \overline{\sigma}E
\]

That is, conservation of mass is the same as conservation of charge, and the linearized momentum equation reduces to a relationship between the current and the electric field through a 3x3 conductivity tensor.
Assuming the magnetic field is oriented along the z axis, the tensor conductivity is:

\[
\bar{\sigma} = \begin{bmatrix}
\sigma_\perp & \sigma_H & 0 \\
-\sigma_H & \sigma_\perp & 0 \\
0 & 0 & \sigma_\parallel
\end{bmatrix}
\]

\[
\sigma_\perp = \frac{(\nu+j\omega)^2 \sigma_\parallel}{(\nu+j\omega)^2 + \omega_{ce}^2}
\]

\[
\sigma_H = \frac{(\nu+j\omega)^2 \omega_{ce} \sigma_\parallel}{(\nu+j\omega)^2 + \omega_{ce}^2}
\]

\[
\sigma_\parallel = \frac{N_e q_e^2 (\nu-j\omega)}{m_e (\nu^2 + \omega^2)}
\]

\[
\omega_{ce} = -\frac{q_e B}{m_e}
\]

Equivalently, we can use the complex permittivity tensor:

\[
\bar{\varepsilon} = \varepsilon_0 \left( I + \frac{\bar{\sigma}}{j\omega} \right)
\]
Dispersion Relation

- Propagation in a medium is described by the dispersion relation, which relates the wavevector \( k \) to the frequency \( \omega \) of a propagating wave.
- To find the dispersion relation, we rewrite Maxwell’s curl equations as:
  \[
  \mathbf{k} \times \mathbf{B} = -\omega \varepsilon_0 \mu_0 \varepsilon_r \mathbf{E} \\
  \mathbf{k} \times \mathbf{E} = \omega \mathbf{B}
  \]
- Substituting to eliminate \( \mathbf{B} \), we have:
  \[
  \mathbf{k} \times \mathbf{k} \times \mathbf{E} + \omega^2 \varepsilon_0 \mu_0 \varepsilon_r \mathbf{E} = 0
  \]
- Rewriting in matrix notation and defining the refractive index vector \( \mathbf{n} = (k c) / \omega \), we have:
  \[
  (\mathbf{n} \mathbf{n}^T - \mathbf{n}^T \mathbf{n} \mathbf{I} + \varepsilon_r) \mathbf{E} = 0
  \]
- A nontrivial solution (\( \mathbf{E} \neq 0 \)) to this system of equations requires:
  \[
  \det (\mathbf{n} \mathbf{n}^T - \mathbf{n}^T \mathbf{n} \mathbf{I} + \varepsilon_r) = 0
  \]
- This dispersion relation works for any medium with a tensor permittivity, not just a plasma.
Dispersion Relation -
The Appleton-Hartree Equation

- By making a few simplifications, we can write an equation for the dispersion relation.

- Choose a coordinate system such that the magnetic field \( B \) is in the \( z \) direction, and the wave is propagating in the \( x-z \) plane.

- Substituting into the dispersion relation and solving the resulting quadratic equation for \( n^2 \), we get the Appleton-Hartree equation:

\[
1 - \frac{n^2}{\omega^2} \pm \sqrt{\left(1 - \frac{\beta_n^2 \sin^2 \theta}{2(\omega^2 - j \nu \omega - \omega_{pe}^2)}\right)^2} + \frac{\omega_{pe}^2 \cos^2 \theta}{\omega^2} \]

Where:

\[
\omega_{ce} = \frac{-\frac{q_e H}{m_e}}{m_e}
\]

\[
\omega_{pe} = \sqrt{\frac{N_e q_e^2}{\epsilon_0 m_e}}
\]
The refractive index

- The refractive index $n$ is, in general, a complex number:
  - If $n$ is purely imaginary, the wave is evanescent.
  - If $n$ is purely real, the wave is propagating.
  - If $n$ is complex, the wave is propagating and attenuated with distance (loss).

Right - refractive indices plotted as a function of angle for propagation at 20 kHz within the earth’s ionosphere, at two separate altitudes.

At 86 km, two modes are propagating. Note the anisotropy.

At 161 km, only one mode (the whistler mode) is propagating. Note the strong anisotropy. Propagation is not possible at angles perpendicular to the magnetic field, where the refractive index goes to infinity.
Reflection coefficients

Sharp ionospheric boundary

Perpendicular incidence

Parallel incidence

Perpendicular reflection

Parallel reflection

\[ R(\theta_i) = \begin{pmatrix} R_{\parallel} & R_{\perp} \\ R_{\parallel} & R_{\perp} \end{pmatrix} \]
- Tenuous nighttime ionosphere
- Reflection coefficient varies by incidence angle
- Determinant of $R$ is < 0, so some signal absorbed/transmitted through ionosphere
- Numerical calculation is in general difficult (not always stable)
- Simple numerical calculation may be less accurate for high incidence angles (propagation well above mode cutoff)
Estimating losses is difficult! Make assumptions:

- Normal incidence
- Wavelength is much smaller than the size of any variation in the medium.

Under these assumptions, the loss is proportional to the imaginary part of the refractive index (the Helliwell approximation):

\[
\alpha = 8.69 \int_{z_1}^{z_2} \Im \{n\} \frac{\omega}{c} \, dz
\]

(Absorption coefficient in dB)

Sample nighttime electron density and collision frequency profiles

Vertical refractive index as a function of altitude.
Estimating Losses - Other Techniques

- Finite difference, finite volume, or finite elements
  - Resource-intensive but always “correct” in the limit as cell size goes to zero.
    - However, FD techniques have difficulty when the losses are high.
  - Examples
    - Chevalier, T. 2006 (FDFD)
    - Lee and Kalluri 1999 (FDTD)

- Full-wave modeling:
  - Divide the domain into layered homogeneous segments
  - Limited applicability but much faster than FD, FV, or FE methods.
  - Solve for the reflection and transmission coefficients at each boundary
  - Examples:
    - Nygren, T. 1982
    - Nygren, T. 1981
    - Lehtinen, N. 2007
References

- Chevalier, T. et al. 2006 (FDFD) - “Terminal impedance and antenna current distribution of a VLF electric dipole in the inner magnetosphere” - IEEE Transactions on Antennas and Propagation, accepted for publication
- Lehtinen, N. et al. 2007 - “Emission of ELF/VLF waves by harmonically varying currents in stratified ionosphere, with application to emission by a modulated electrojet” (Submitted to Geophysical Research Letters, in review)