A New Approach for Evaluating Clipping Distortion in Multicarrier Systems

Ahmad R.S. Bahai, Manoneet Singh, *Student Member, IEEE,* Andrea J. Goldsmith, and Burton R. Saltzberg, *Fellow, IEEE*

Abstract— **Multicarrier signals are known to suffer from a high peak-to-average power ratio (PAPR), caused by the addition of a large number of independently modulated subcarriers in parallel at the transmitter. When subjected to a peak-limiting channel, such as a non-linear power amplifier, these signals may undergo significant spectral distortion, leading to both in-band and out-of-band interference, and an associated degradation in system performance. This paper characterizes the distortion caused by the clipping of multicarrier signals in a peak-limiting (nonlinear) channel. Rather than modeling the effects of distortion as additive noise, as is widespread in the literature, we identify clipping as a rare event and focus on evaluating system performance based on the conditional probability of bit error given the occurrence of such an event. Our analysis is based on the asymptotic properties of the large excursions of a stationary Gaussian process, and offers important insights into both the true nature of clipping distortion, as well as the consequent design of schemes to alleviate this problem.**

Keywords— **Peak-to-Average Ratio, OFDM, Clipping.**

I. Introduction

RECENTLY, there has been much interest in using multicarrier modulation in a wide variety of application scenarios [1]. The most well-known ones are DMT (Discrete Multitone) in the wired ADSL (asymmetric digital subscriber line) and VDSL (very high-rate digital subscriber line) domain [2], and OFDM, or orthogonal frequency division multiplexing, for a variety of wireless applications such as DAB (digital audio broadcasting), DVB-T (digital video broadcasting- terrestrial), Wireless LANs, MANs, and high-speed cellular data [3]. In all these systems, information is transmitted in parallel on several, orthogonal subcarriers, each subcarrier being QAM/ PSK modulated. As the number of subcarriers N becomes large, the multicarrier (OFDM) baseband signal can be quite accurately represented as a complex Gaussian process with Rayleigh envelope distribution and uniform phase distribution [4],[5]. The Gaussian approximation is a consequence of the Central Limit Theorem and is very realistic for most practical systems ($N \geq 64$) of interest.

A major implication of the Gaussian nature of the composite multicarrier waveform is the great variability of its instantaneous envelope, which depends on the phase of each composing subcarrier at any specific instant. In the most extreme case, all the different subcarriers may add up in phase, producing an envelope peak equal to the sum of amplitudes of the individual subcarriers. These amplitude spikes then make the multicarrier signal very sensitive to distortion by non-linear devices, such as A/D converters, IFFT/FFT processors with finite word length, RF power amplifiers, etc. Since the linear operation of all these devices over a large dynamic range entails considerable cost and power penalties, a significant body of recent research has focussed on achieving low multicarrier PAPR via various distortionless and distortion-inducing techniques (see, for example, [6]-[8], and references therein). At the same time, the out-of-band interference generated by the clipping of OFDM signals has served as one of the major obstacles against the adoption of OFDM as standard for several emerging wireless applications [1]. An accurate characterization of the effects of clipping distortion is therefore critical to the future of multicarrier techniques. This paper offers such a characterization.

Conventional analysis of the effects of clipping on a multicarrier signal treats the distortion caused by clipping as additive Gaussian noise, with variance equal to the energy of the clipped portion of the composite waveform [9]-[16]. This approach is reasonable if the clipping level is set sufficiently low to produce several clipping events during an OFDM symbol interval. In most realistic cases, however, particularly when the desired error probability is low, the clipping level is set high enough such that clipping is a rare event, occurring more infrequently than once every symbol interval. Clipping under these conditions then forms a kind of impulsive noise rather than a continual background noise, leading to a very different type of error mechanism. Besides, the conventional additive noise model offers no spectral characterization of the nature of the distortion, *i.e.*, the interference between adjacent bands that arises as a result of clipping. An alternate approach presented by some authors [17], [18] seeks to factor in the effect of this spectral leakage by considering the power spectral density (PSD) of the distortion component (which can be computed, for example, by taking the Fourier transform of its autocorrelation function [18]). However, note that any technique that involves such spectral estimation would imply an inherent statistical averaging. Therefore, once again, such an estimation cannot accurately reflect the instantaneous or impulsive nature of the clipping process.

In this paper, we develop a novel approach for analyzing the effects of clipping on an unconstrained multicarrier signal. The approach explicitly identifies clipping as a rare event, and is based on the properties of the large

Ahmad Bahai is with National Semiconductor, Fremont, CA, USA. Manoneet Singh was with the Dept. of Electrical Engineering, Stanford University, Stanford, CA-94305, USA. He is now with Radia Communications, Sunnyvale, CA-94086, USA. Andrea Goldsmith is with the Dept. of Electrical Engineering, Stanford University, Stanford, CA-94035, USA. Burton Saltzberg is a consultant at 91 Southview Terrace, Middletown NJ-07748, USA.

deviations of a stationary Gaussian process. For purposes of analysis, we focus on an OFDM system that performs an Inverse Discrete Fourier Transform (IDFT) of the N tone complex data vector $\vec{X} = \{X_k\} = [X_0 \cdots X_{N-1}]^T$, and transmits the time-domain vector of length N, $\vec{x} = \{x_n\} = [x_0 \cdots x_{N-1}]^T = IDFT(\vec{X})$. As mentioned earlier, for large N, the output time vector distribution can be well approximated by a vector of uncorrelated complex truncated Gaussian samples. The PAPR is then defined to be

$$PAPR(\vec{x}) = \frac{||\vec{x}||_\infty^2}{E(||\vec{x}||_2^2)/N} \qquad (1)$$

where $E(.)$ denotes the expectation, and $||\vec{x}||_\infty$ and $||\vec{x}||_2$ represent the ∞-norm and 2-norm of $\vec{x}$ respectively. This PAPR may be as large (for PSK modulation) as N^2/N when all the N subchannels add coherently, i.e., in phase. However, as several authors have pointed out [5],[6], the probability of co-phasing N complex independent variables is indeed very small, even for moderate values of N. The absolute peak is thus not an appropriate measure for most practical applications. A more reasonable measure would be to define the "peak" of the signal power such that the probability of crossing that level is negligible. For example, a threshold of 5.2σ ($||.||_2^2 = \sigma^2$ = average power) would correspond to a crossing probability of 10^{-7}. Clipping would thus occur whenever the instantaneous signal envelope exceeds 5.2 times the average (rms) power. Such a definition of peak-to-average ratio, unlike the absolute maximum, would then be independent of the number of subcarriers.

The remainder of this paper is organized as follows. In Section 2, we briefly review some asymptotic properties of stationary Gaussian processes that are relevant to our analysis. Section 3 presents the basic system model, and computes a bound on the BER degradation due to clipping of a multicarrier signal. Section 4 analyzes performance in AWGN and fading channels, with simulation results for both Rayleigh and multipath fading environment. Section 5 presents the major conclusions.

II. Background

We will be concerned with stationary Gaussian processes, possessing finite second-order moments, and continuous with probability one. The following *three* asymptotic properties of the large excursions of such processes will form the basis of our analysis in the sequel. These properties are originally due to Rice [19] and have also been elaborated on or discussed by several subsequent authors (c.f. [20]-[24]). They are depicted conceptually in Fig. 1.

- The sequence of upward level crossings [1] of a stationary and ergodic process, having continuous sample function with probability one, asymptotically approaches a Poisson process for large levels l. For a Gaussian signal $x(t)$, the rate of this Poisson process is given by [23]

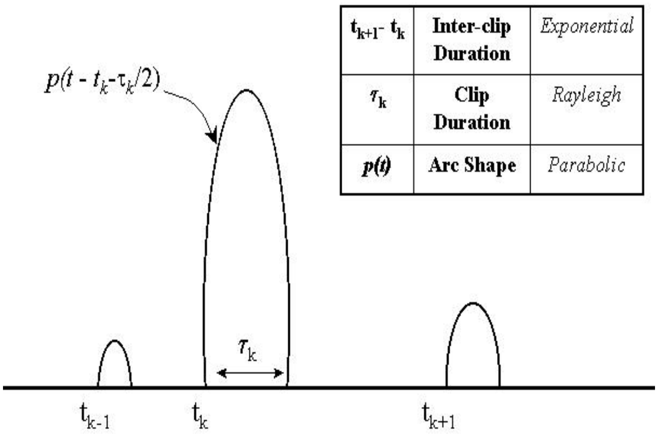

Fig. 1. Excursions of a Gaussian random process above $|l|$

$$\lambda_l = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}}\exp\frac{-l^2}{2m_0}, \qquad (2)$$

where

$$m_i \doteq \int \omega^i dF(\omega), i = 0, 2, \cdots, \qquad (3)$$

and $dF(\omega)$ is the power spectral density of $x(t)$. Without loss of generality, we can normalize the power m_0 in the signal $x(t)$ to unity. The relevant quantity m_2 then represents the power in the first derivative of $x(t)$.

- The length of intervals τ during which the signal stays above the high level $|l|$ is (asymptotically) Rayleigh distributed [21], with density function

$$\rho_\tau(\tau) = \frac{\pi}{2}\frac{\tau}{\tau_m^2}e^{-\frac{\pi}{4}(\frac{\tau}{\tau_m})^2}, \tau \geq 0. \qquad (4)$$

Here τ_m denotes the expectation of τ. Since $\lambda_l \tau_m = Pr(x(t) \geq l)$, the expected value of the duration of a clip may be approximated as [21]:

$$\tau_m = \frac{Q(l)}{\lambda_l} \approx \frac{\sqrt{2\pi}}{l\sqrt{m_2}} \qquad (5)$$

where

$$\begin{aligned} Q(z) &= \frac{1}{\sqrt{2\pi}}\int_z^\infty e^{-u^2/2}du \\ &= \frac{1}{\sqrt{2\pi}z}e^{-z^2/2}(1 - \frac{1}{z^2} + \frac{3}{z^4} - \cdots) \qquad (6) \end{aligned}$$

The approximation in (5) is valid for $l > 3\sigma$.

- The shape of pulse excursions above level $|l|$ are parabolic arcs [19],[22] of the form

$$p_\tau(t) = (-\frac{1}{2}m_2 t^2 + \frac{1}{8}m_2\tau^2)l.\ \mathrm{rect}(\frac{t}{\tau}), \qquad (7)$$

[1] A random process $x(t)$ is said to have an upcrossing of the level l at time t_0 if $\exists(\epsilon > 0) : x(t) \leq l$ for $t \in (t_0 - \epsilon, t_0)$, and $x(t) > l$ for $t \in (t_0, t_0 + \epsilon)$.

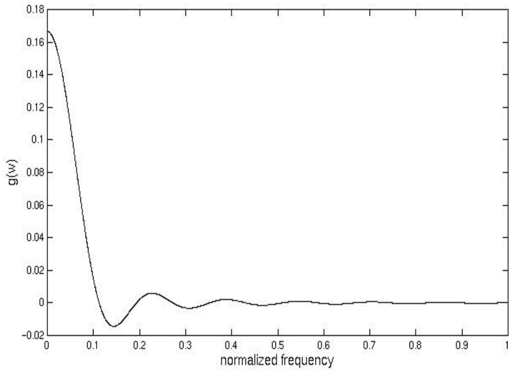

Fig. 2. Normalized instantaneous distortion spectrum

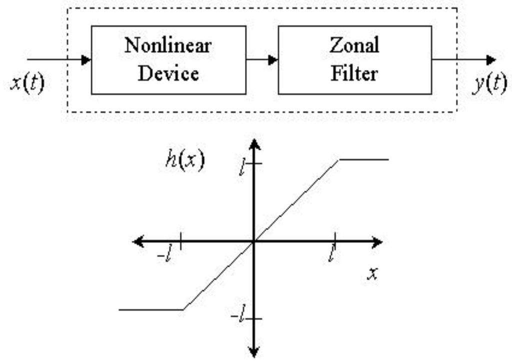

Fig. 3. Memoryless nonlinear mapping

where rect(.) denotes a rectangular window function, and τ, the (random) duration of the clip, forms the support of the parabolic arc (Fig. 1). The parabolic shape of the signal's crossings above high level l follows naturally from a Taylor's series approximation about the peak value (cf. [24], and references therein). We will also require the *instantaneous* spectrum of the clipped signal component for subsequent analysis. This can be obtained by taking the Fourier transform of the parabolic arc (7), to get

$$g_\tau(\omega) = m_2 \frac{l\tau}{\omega^2}(\text{sinc}\frac{\omega\tau}{2} - \cos\frac{\omega\tau}{2}), \quad (8)$$

which is plotted in Fig. 2 ($\text{sinc}\,\varphi \doteq \sin\varphi/\varphi$). Note that the instantaneous spectrum is not flat over all frequencies. We shall have more to say about this fact in the following sections.

III. Error Probability Analysis

In this section, we will compute a bound on the bit error floor caused by clipping in a multicarrier system (in the absence of noise and other channel impairments). The basic system model used for this analysis is presented next.

A. System Model

Consider a continuous time, unconstrained, baseband multicarrier signal $x(t)$, input to a soft (clipping) nonlinearity $h(.)$, to produce the (clipped) output waveform $y(t)$ (Fig. 3). The non-linear transfer function may be described, e.g.,

$$y = h(x) = \begin{cases} -l & x \leq l; \\ x & |x| < l; \\ l & x \geq l. \end{cases} \quad (9)$$

where explicit dependence on time is omitted from the notation above. For a memoryless nonlinearity such as (9), the output $y(t)$ may be decomposed by Bussgang's theorem [25] into two *uncorrelated* signal components

$$y(t) = \gamma x(t) + c_L(t), \quad (10)$$

where the value of γ that uncorrelates the "error" $c_L(t)$ and input signal $x(t)$ is given by

$$E[x(t+\tau)y(t)] \quad = \quad \gamma E[x(t+\tau)x(t)], \quad (11)$$

from which it is straightforward to show that $\gamma \approx 1$ for $l \gg 1$. We can thus use a linear model to analyze the effect of clipping on the transmitted signal.

Under the assumption of many subcarriers, the power spectrum of the $x(t)$ will tend to rectangular over a total frequency $f_0 = N/T$ (T = OFDM symbol duration) [1]. Hence, the rate of the Poisson process corresponding to the clipping of $x(t)$ can be defined from (2) as:

$$\lambda_l = \frac{1}{2\pi}\sqrt{m_2}\exp\frac{-l^2}{2} = \frac{f_0}{\sqrt{3}}\exp\frac{-l^2}{2}. \quad (12)$$

Also, the expected value for the duration of a clip will be given (from (5)) by

$$\tau_m^{-1} \approx \sqrt{\frac{2\pi}{3}} f_0 l. \quad (13)$$

B. BER due to clipping

Having defined the relevant parameters, we now turn to evaluating the metric of interest in this paper, which is a conditional rate of symbol error. Define A as the event of clipping a signal above level $|l|$ in any given symbol interval (and A^c as its complement). Then, by the law of total probability

$$\begin{aligned} Pr(\text{symbol error}) \quad &= Pr(\text{symbol error}|A)Pr(A) \\ &+ Pr(\text{symbol error}|A^c)Pr(A^c). \quad (14) \end{aligned}$$

In order to compute this probability, we will first evaluate the effect of a clip on each of the individual subchannels. The effect of a clip of duration τ, occurring at time t_0, on the kth sub-channel is given by the Fourier Transform of the clipped portion of the pulse,

$$F_k = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} f_n e^{\frac{-j2\pi kn}{N}}, \quad (15)$$

where f_n are samples of the clipped pulse $p(t)$ in (7)

$$f_n = p(t - t_0 - \tau/2)|_{t=\frac{nT}{N}}, \quad (16)$$

and the factor $1/\sqrt{N}$ preserves total power. We can replace the discrete Fourier Transform with the conventional one by substituting

$$f_n = \frac{1}{\delta} \int_{n\delta-\delta/2}^{n\delta+\delta/2} p(t)dt, \quad (17)$$

$\delta = T/N$. This substitution is valid since $\delta \ll T$. Then

$$F_k = \frac{N}{T} \frac{1}{\sqrt{N}} \int_{t_0}^{t_0+\tau} p(t - t_0 - \tau/2) e^{\frac{-j2\pi kt}{T}} dt. \quad (18)$$

Substituting $u = t - t_0$ in (18) yields

$$\begin{aligned} F_k &= \tfrac{\sqrt{N}}{T} e^{-\frac{j2\pi kt_0}{T}} \int_0^\tau p(u - \tau/2) e^{\frac{-j2\pi ku}{T}} du \\ &= \tfrac{\sqrt{N}}{T} e^{-\frac{j2\pi kt_0}{T}} g(\tfrac{2\pi k}{T}), \end{aligned} \quad (19)$$

where $g(.)$ is pulse spectrum given by (8). Thus,

$$F_k = \frac{\sqrt{N} m_2 T l \tau}{4\pi^2 k^2} e^{-\frac{j2\pi kt_0}{T}} (\text{sinc}(\frac{\pi k\tau}{T}) - \cos(\frac{\pi k\tau}{T})). \quad (20)$$

For $\alpha \ll 1 (\tau \ll T)$, we can use the expansion

$$\text{sinc}(\alpha) - \cos(\alpha) \approx \frac{\alpha^2}{3} \quad (21)$$

in (20) in order to approximate the response of the kth subcarrier to a clip of duration τ as

$$F_k = \frac{1}{12T} \sqrt{N} m_2 l \tau^3 \exp j\theta. \quad (22)$$

In (22), θ is uniformly distributed over $[0, 2\pi]$, and the Rayleigh probability distribution of τ is given by $\rho_\tau(\tau)$ in (4). Note that the expansion in (21) will not be valid in general for the higher frequency subcarriers in the multicarrier signal. In fact, the nature of the distortion spectrum in (19) would indicate that the probabilities of error due to a clip will vary across subcarriers, the overall probability being dominated by that of the lower subcarriers[2]. We can, however, use the response in (22) to compute an upper bound on the error probability due to clipping. Rewriting equation (22) as

$$F_k = \eta \exp j\theta, \quad (23)$$

we are interested in obtaining the probability distribution of η:

$$Pr(\eta > R) = Pr[\tau > (\frac{12RT}{\sqrt{N} m_2 l})^{\frac{1}{3}}]. \quad (24)$$

[2]This would be true if the constellation sizes on the different subcarriers are equal, or even more so when the lower subcarriers carry larger constellations, as is typically the case in ADSL (due to bit loading [2]).

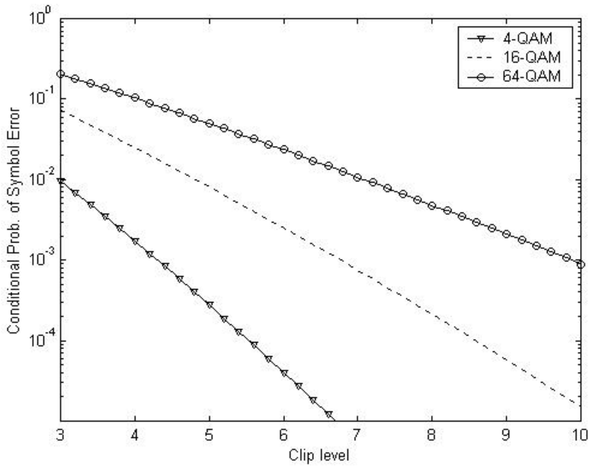

Fig. 4. Conditional probability of symbol error.

Using the distribution of τ in (4), and substituting for m_2 from (3), we get, after some simplification:

$$Pr(\eta > R) = \exp -[\frac{3\pi^2 R^2 l^4 N}{8}]^{\frac{1}{3}}. \quad (25)$$

If we warp the complex plane by the mapping $a \exp j\theta \rightarrow a^{\frac{1}{3}} \exp j\theta$, then (25) would represent a Rayleigh distribution in the warped complex plane. The real or imaginary component of F_k therefore has a normal distribution with

$$Pr(\eta \cos\theta > x) = Q(\frac{x^{\frac{1}{3}}}{\sigma}), \quad (26)$$

where $\sigma \doteq (\frac{2}{\sqrt{3N}\pi l^2})^{\frac{1}{3}}$.

To compute the error probability, we will assume that each subcarrier carries a square constellation of L^2 points. Each (real/imaginary) component has L levels, equally spaced and separated by $2d$, with total power equal to $1/2N$ (since we have normalized the total power to unity). Thus

$$d = \sqrt{\frac{3}{2N(L^2 - 1)}}. \quad (27)$$

Combining (25) and (27), we get the probability of error in any particular subcarrier, given that a clip occurs, as:

$$\begin{aligned} Pr(error|A) &= \tfrac{4(L-1)}{L} Q(\tfrac{d^{\frac{1}{3}}}{\sigma}) \\ &= \tfrac{4(L-1)}{L} Q([\tfrac{3\pi l^2}{\sqrt{8(L^2-1)}}]^{\frac{1}{3}}). \end{aligned} \quad (28)$$

Note that this quantity is independent of N, the total number of subcarriers. This is because although the effect of a clip increases with the number of subcarriers (equation 22), the signal power in each component is inversely proportional to N. Also, not only does the probability of occurrence of a clip decrease with the clipping level (equation (12)), but so does the probability of error caused by each occurring clip. Equation (28) is plotted in Fig. 4 for different constellation sizes L^2.

To compute the overall probability of error, we will need to multiply the conditional probability in (28) with the probability of occurrence of a clip in one symbol duration. This can be obtained using (2), the Poisson rate of clipping events, as

$$Pr(clip) = 1 - e^{\lambda_l T} \approx \lambda_l T. \tag{29}$$

For 2 sided clipping we will have

$$Pr(A) = 2\lambda_l T. \tag{30}$$

so that the overall probability of symbol error is upper bounded by

$$Pr(error) = \frac{8N(L-1)}{\sqrt{3}L} e^{-\frac{l^2}{2}} Q([\frac{3\pi l^2}{\sqrt{8(L^2-1)}}]^{\frac{1}{3}}). \tag{31}$$

In particular, for $L = 2$ (QPSK), which is typical in several wireless OFDM systems, the error probability is extremely small, being given by

$$Pr(error) = \frac{4N}{\sqrt{3}} e^{-\frac{l^2}{2}} Q([\sqrt{\frac{3}{8}}\pi l^2]^{\frac{1}{3}}). \tag{32}$$

On the other hand, for large constellation sizes, the error probability may be approximated as

$$Pr(error) = \frac{8N}{\sqrt{3}} e^{-\frac{l^2}{2}} Q([\frac{3\pi l^2}{\sqrt{8L}}]^{\frac{1}{3}}). \tag{33}$$

The Poisson probability in (29) would hold for a continuous time signal, *i.e.,* an infinitesimal sampling of the multicarrier waveform. Since practical transceivers operate at Nyquist frequency (or oversampled versions of this frequency), we can substantially tighten the error bound in (31) by replacing this clip probability with its corresponding discrete time value. Specifically, the probability of a single sample exceeding the level $|l|$ is $2Q(l)$, so assuming the samples to be i.i.d.[3], the probability of clipping m samples within an OFDM symbol will be given by the Binomial distribution

$$\begin{aligned} p_m &= \begin{pmatrix} N \\ m \end{pmatrix} (1-2Q(l))^{N-m}(2Q(l))^m \\ &\approx 0 \text{ for } l \gg 1 \text{ and } m > 1, \end{aligned} \tag{34}$$

and $p_m \approx 2NQ(l)$ for $m = 1$. The error probability in (31) then becomes

$$Pr(error) = \frac{8N(L-1)}{L} Q(l).Q([\frac{3\pi l^2}{\sqrt{8(L^2-1)}}]^{\frac{1}{3}}). \tag{35}$$

This probability $P(error)/N$ is plotted in Fig. 5 for various typical constellation sizes L^2.

It is interesting to compare these results with those obtained using the "additive Gaussian noise" model, which

[3]Since the IDFT of statistically independent, Gaussian spectral points produces statistically independent time domain samples; cf. [5],[6].

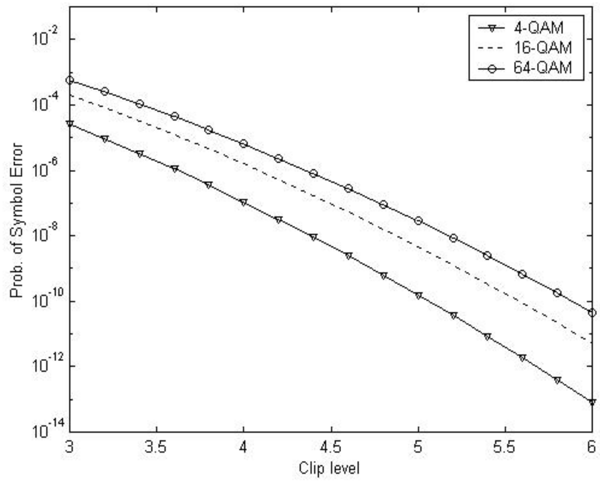

Fig. 5. Symbol error probabilities due to clipping

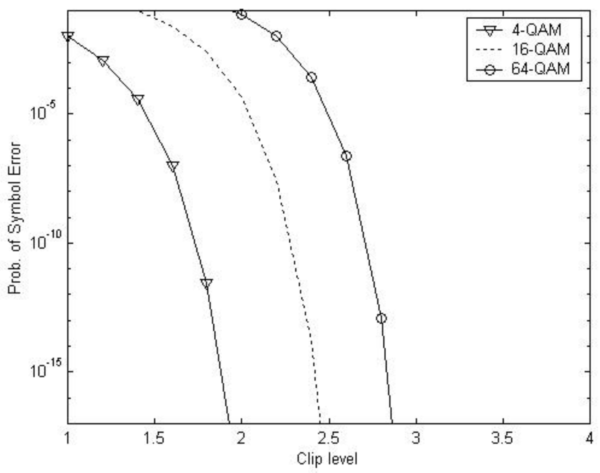

Fig. 6. Symbol error probabilities using the additive Gaussian noise approach

would hold if the clip level were set low enough to permit several clipping events per OFDM symbol. In reference [9], the power in the clipped portion $c_L(t)$ of the input waveform $x(t)$ has been computed as

$$\begin{aligned} \sigma_c^2 &= 2\int_l^\infty (x-l)^2 e^{-x^2/2} dx \\ &= -\sqrt{\tfrac{2}{\pi}} l \exp -\tfrac{l^2}{2} + 2(1+l^2)Q(l), \end{aligned} \tag{36}$$

where we have normalized the result to unity signal power. This result is pessimistic since it does not account for the fact that some of the noise power will fall out-of-band for the OFDM band of interest, and will therefore not cause any in-band distortion. Nevertheless, for the (Gaussian) noise power level in (36), the probability of error for any

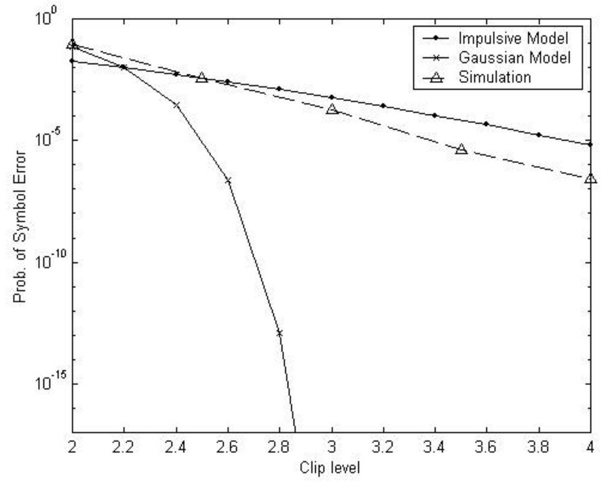

Fig. 7. Simulated and analytical symbol error probabilty floor due to clipping

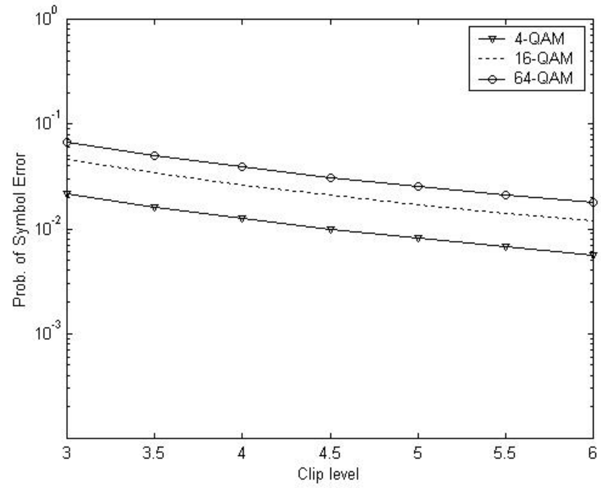

Fig. 8. Symbol error probability floor in a Rayleigh fading channel

one subcarrier would be given by

$$Pr(error) = 4\frac{L-1}{L}Q(\frac{\sqrt{3}}{\sigma_c\sqrt{(L^2-1)}}). \quad (37)$$

This probability is plotted in Fig. 6. It is easy to see from this curve that the error probabilities as computed by the additive noise model are substantially lower than those obtained using our analysis, even under the pessimistic approximation described above. This is because instead of being spread uniformly over time, as is assumed in [9]-[16], noise due to clipping is in fact concentrated in time as impulses, leading to a greater probability of error over the various subcarriers.

The validity of the two models discussed in this section was confirmed by simulating the error probability of a 64-tone OFDM signal modulated using a 64-QAM constellation, soft clipped at various levels of output amplitude. The results, shown in Fig. 7, suggest that while the additive noise model is reasonable for low clipping thresholds, it underestimates the error probability by several orders of magnitude for the higher clip levels. This implies that we must account for the impulsive nature of the clipping process in estimating error probabilties for high clip thresholds.

We can also use equation (35) to compute an approximate bound on the probability of error *per bit.* Assuming Gray coded levels in each subcarrier component, and noting that errors beyond adjacent levels are negligible, we get the probability of bit error

$$Pr_b(error) \approx \frac{4N(L-1)}{L\log_2 L}Q(l).Q([\frac{3\pi l^2}{\sqrt{8(L^2-1)}}]^{\frac{1}{3}}). \quad (38)$$

Note that this value will also hold in the presence of fading, as long as the fading is slow compared to the OFDM symbol rate, and the clip occurs at the *transmitter.* In practice, however, we are mostly concerned with bit error rate performance of the OFDM *receiver.* The analysis for clipping at the receiver (in the presence of channel impairments) is presented in the next section.

IV. Performance in AWGN and Fading

Consider first the case when additive white Gaussian noise of (random) amplitude n is also present on the channel. In this case, the per-bit error probability in a lower subcarrier component would be (from (26))

$$\begin{aligned} &Pr_b(error|clip, noise) \\ &= \tfrac{2(L-1)}{L\log_2 L}\int Pr(rcos\theta > d-n)p_N(n)dn \\ &= \tfrac{2(L-1)}{L\log_2 L\sqrt{2\pi}\sigma_n}\int Q[\tfrac{(d-n)^{\frac{1}{3}}}{\sigma}]e^{\frac{-n^2}{2\sigma_n^2}}dn \end{aligned} \quad (39)$$

where σ_n^2 is the power (variance) of the AWGN, and the effect of the AWGN is to reduce the effective distance between various constellation points (from d to $d-n$). From (14), the overall error probability is upper bounded by

$$\begin{aligned} Pr_b(error) &= \tfrac{2(L-1)}{L\log_2 L} \\ (\int \tfrac{\sqrt{2}NQ(l)}{\sqrt{\pi}\sigma_n}Q[\tfrac{(d-n)^{\frac{1}{3}}}{\sigma}]&\exp(-\tfrac{n^2}{2\sigma_n^2})dn + Q(\tfrac{d}{\sigma_n})). \end{aligned} \quad (40)$$

To examine the effect of fading on these results, consider a noiseless, flat-fading channel, modeled by a slowly-varying Rayleigh random variable with long-term distribution function

$$p_Z(z) = \frac{\pi}{2}ze^{-\frac{\pi z^2}{4}}, z > 0, \quad (41)$$

where the mean has been normalized to unity, without loss of generality. We assume the channel to be constant over one OFDM symbol duration. The error probability for a symbol during which the channel amplitude variable is z is then

$$Pr(error|z) \approx 8\frac{N(L-1)}{L}Q(lz)Q([\frac{3\pi l^2 z^2}{\sqrt{8(L^2-1)}}]^{\frac{1}{3}}). \quad (42)$$

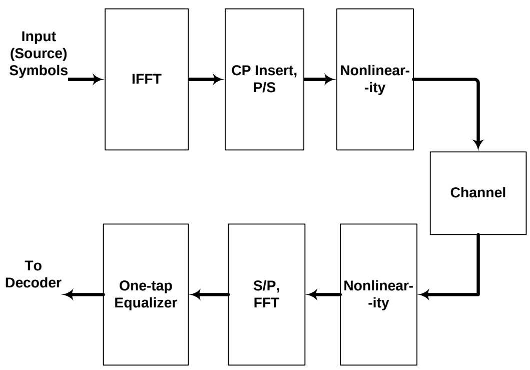

Fig. 9. Baseband equivalent system model for clipping at OFDM receiver

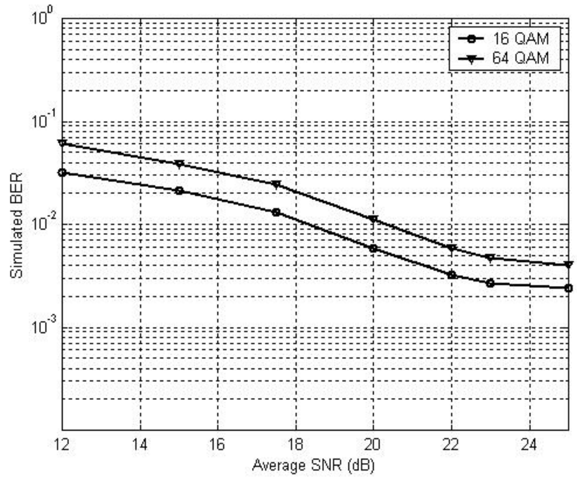

Fig. 10. BER in a Rayleigh fading channel, clip level=3σ

Thus the overall error probability in this case is upper-bounded by

$$Pr(error) = 4\pi \frac{N(L-1)}{L} \int_0^\infty z e^{\frac{-\pi z^2}{4}} Q(lz) Q\left(\left[\frac{3\pi l^2 z^2}{\sqrt{8(L^2-1)}}\right]^{\frac{1}{3}}\right) dz, \tag{43}$$

which can be expressed for purposes of numerical computation in definite integral form as

$$Pr(error) = \frac{4N(L-1)}{\pi L} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^\infty z \exp\left(\frac{-z^2}{4}\left[\frac{\pi}{2} + \frac{2l^2}{\sin^2\varphi} + \frac{\xi^{\frac{1}{3}}}{\sin^2\psi}\right]\right) d\varphi \, d\psi \, dz, \tag{44}$$

where

$$\xi \doteq \frac{9\pi^2 l^4}{z^2(L^2-1)}. \tag{45}$$

Note that the alternate representation of the Q function [26] has been used to derive (44) from (43). This error probability (in (44)) is computed numerically and plotted in Fig. 8. It is clear from this figure that the performance with fading is far worse than that obtained in the absence of fading, discussed in the previous section. Therefore, it appears that while the OFDM signal is relatively insensitive to clipping at the transmitter, the performance is somewhat degraded in the presence of channel fades, together with clipping at the receiver. Of course, (43) only represents an upper bound on the probability of error due to clipping, that is tight for lower subcarriers of the multicarrier signal.

In order to validate the analytical results presented in this section, as well as to generalize these results for the case of a multipath fading channel, extensive, *bit-true* simulations were performed using the OFDM baseband equivalent system model shown in Fig. 9. A 64-point FFT signal with 52 "active" tones was simulated, and a guard interval of length 16 was added to each OFDM symbol as cyclic prefix to compensate for ISI. Two channels were considered: (i) a Rayleigh (no diversity) channel modeled using a single "tap" of average power 0 dB, corresponding to a severe, flat fading environment; and (ii) a three-path WSSUS multipath channel with an exponentially decaying power profile and rms delay spread of 100 nsec, corresponding to a typical indoor environment [1, pp.183]. Perfect synchronization and a one-tap, post-FFT equalizer with perfect channel estimates were assumed in order to observe the effect of the clips alone. In each case the bit error probability was recorded as a function of the received signal SNR, averaged over a large number (10^4) of fading instances. The resulting error probability curves are presented in Figs. 10 through 13.

Several observations can be made from these curves. First, as expected, the bit error performance due to clipping deteriorates with increasing constellation size L^2, and improves as the clip level is increased. Second, the performance in Rayleigh (flat) fading is observed to be consistently worse than that seen in AWGN, for a given clip level and constellation size. Also, a comparison of Figs. 10 and 11 with the analytical curves in Fig. 8 reveals that the residual symbol error floor due to clipping (even at the high SNRs) is within an order of magnitude of the theoretical bound computed in equation (43).

Other results, not shown here, also indicated that the BER performance was not equal on all subcarriers; the lower subcarriers being subject to about 1 to 1.5 dB greater degradation than those at the extremes. This is in accordance with our observation about the nature of the instantaneous clipping spectrum in Section II. The error floor for the "worst" affected subcarrier was calculated and used to compare with the analytical bound for error probability described in equation (43). As evident from Figs. 8 and 14, the theoretical and simulated results are in good agreement for the higher clip levels, where the conditions discussed in Section II for the rare event model are satisfied. This shows that the derived analytic bound is tight for lower subcarriers of the multicarrier signal.

Apart from the above discussion about error performance, there are several other practical (implementation)

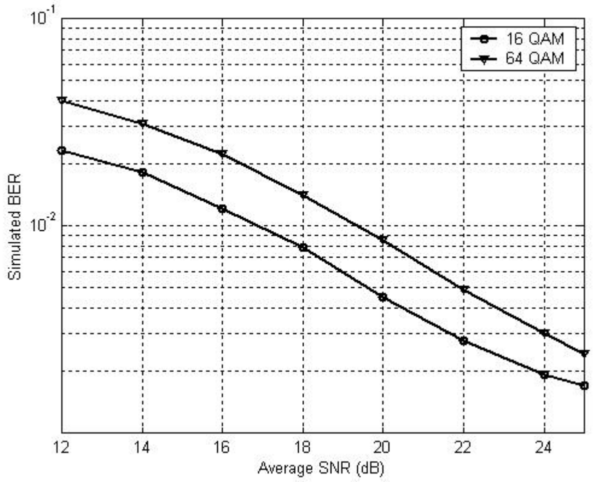

Fig. 11. BER in a Rayleigh fading channel, clip level=6σ

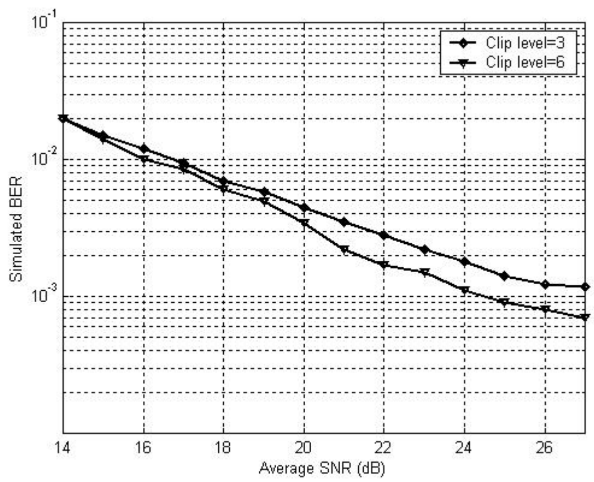

Fig. 12. BER for uncoded 16-QAM in a multipath fading channel

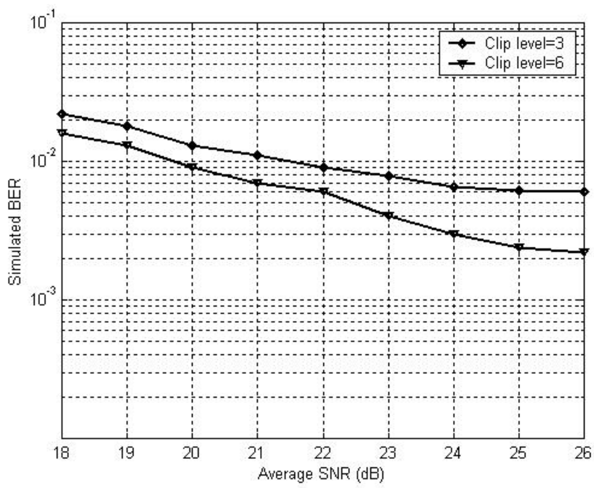

Fig. 13. BER for uncoded 64-QAM in a multipath fading channel

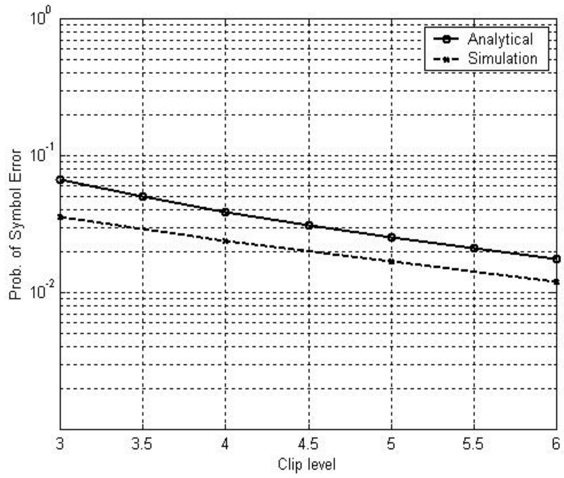

Fig. 14. Comparison of simulated and analytical symbol error probability floor for 64-tone OFDM signal in Rayleigh fading

issues associated with clipping at the receiver as well. For example, in a packet communication system, the AGC and clip level are set during the training phase or "preamble" preceding a data packet; consequently the peak variations (*e.g.*, caused by fading) in the packet itself are not known a priori to the receiver. The choice of an appropriate clip level is therefore complicated and involves a tradeoff between excessive clipping noise on the one hand, and a dominant A/D quantization noise floor on the other. Also, even though most analysis, including the one in this paper, deals with memoryless nonlinearities, real devices have some memory that arises when various circuit elements are driven into saturation. This memory would only worsen the error performance in the presence of clipping.

A final, critical issue for wireless systems that we have not addressed in this paper is that of Out-of-Band (OOB) interference (or *spectral regrowth*) caused by the clipping of multicarrier signals. Since OOB power in wireless systems is generally subject to a regulatory limit, clipping has the effect of reducing the mean envelope power allowed under OFDM, as compared to that allowed under a constant envelope modulation. Alternatively, power amplifiers at the transmitter must be designed to behave linearly up to the *peak* envelope power, which requires considerable backoff, thus increasing cost and reducing overall efficiency of operation. In either case, the high envelope variability of the OFDM waveform seems to pose a significant challenge to designers of power-constrained wireless applications.

Two issues need to be considered in evaluating OOB effects in multicarrier signals. The first is to compute the average (long-term) power spectral density (PSD) of the clipped OFDM signal, to determine whether the signal meets regulatory requirements. This issue is relatively straightforward and has been addressed by several authors; see, for example, [12],[13],[20],[22], and references therein.

A second issue is to ascertain the *actual* effect that clipping would have on the BER of a user in an adjacent OFDM band. This issue is more complicated, however, since the bursty nature of the clips needs to be taken into account in order to observe the conditional rate of bit errors due to clipping. Otherwise, instantaneous power spill is averaged to a smaller value and may mislead system designers. To analyze the effect of clipping on the performance of a receiver operating on an adjacent band, we can assume another similar OFDM system, occupying the adjacent channel band with a suitable (frequency) guard interval. Then, following Section III, F_k in equation (20) describes the effect of a clip on subchannel k. However, we are now interested in an adjacent system, therefore $k > N$, and the approximation used in (21) is no longer valid. Also, F_k needs to be multiplied by H_k, the value of the composite filtering function at a particular subcarrier frequency[4]. Since F_k and H_k are in general probabalistic quantities, either numerical techniques or simulation is required to evaluate the probabilty of error in an adjacent band for a given modulation format and FFT size. This evaluation forms the subject of future work.

V. Conclusions

This paper presents a novel analysis of the effects of clipping in a multicarrier system. In particular, we derive bounds on the probability of error due to clipping at both the transmitter, as well as at the receiver, in presence of channel impairments. Our analysis leads us to conclude that the BER obtained using a "traditional" analysis of clipping distortion, which neglects the impulsive nature of clips, may be optimistic by several orders of magnitude, depending on the constellation size, number of subcarriers, and clip level.

We close with two comments. First, nonlinear distortion due to clipping causes some interference both inside and outside the given signal bandwidth. In many applications, the out-of-band emissions may be intolerable, even if the in-band BER degradation is acceptable or insignificant. Second, there are significant cost and power advantages in being able to reduce the operating dynamic range of various circuit devices. The design of multicarrier waveforms with low peak-to-average ratio therefore remains a subject of enormous academic as well as practical interest.

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